

**JACOB'S LADDERS, REVERSE ITERATIONS AND NEW
INFINITE SET OF L_2 -ORTHOGONAL SYSTEMS GENERATED
BY THE RIEMANN $\zeta(\frac{1}{2} + it)$ -FUNCTION**

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ABSTRACT. It is proved in this paper that continuum set of L_2 -orthogonal systems generated by the Riemann zeta-function on the critical line corresponds to every fixed L_2 -orthogonal system on a fixed segment. This theorem serves as a resource for new set of integrals not accessible by the current methods in the theory of the Riemann zeta-function.

Dedicated to the 100th anniversary of G.H. Hardy's fundamental theorem: the function $\zeta(\frac{1}{2} + it)$ has an infinite set of zeros, [1].

1. INTRODUCTION

1.1. In this paper we obtain new properties of the signal

$$(1.1) \quad \begin{aligned} Z(t) &= e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + it\right), \\ \vartheta(t) &= -\frac{t}{2} \ln \pi + \operatorname{Im} \ln \Gamma\left(\frac{1}{4} + i\frac{t}{2}\right), \end{aligned}$$

which is generated by the Riemann zeta-function. In connection with (1.1) we have introduced (see [5], (9.1), (9.2)) the formula

$$(1.2) \quad \tilde{Z}^2(t) = \frac{d\varphi_1(t)}{dt},$$

where

$$(1.3) \quad \begin{aligned} \tilde{Z}^2(t) &= \frac{Z^2(t)}{2\Phi'_\varphi[\varphi(t)]} = \frac{|\zeta(\frac{1}{2} + it)|^2}{\omega(t)}, \\ \omega(t) &= \left\{1 + \mathcal{O}\left(\frac{\ln \ln t}{\ln t}\right)\right\} \ln t. \end{aligned}$$

The function $\varphi_1(t)$ that we call Jacob's ladder (see our paper [2]) according to the Jacob's dream in Chumash, Bereishis, 28:12, has the following properties:

(a)

$$\varphi_1(t) = \frac{1}{2}\varphi(t),$$

(b) function $\varphi(t)$ is solution of the non-linear integral equation (see [2], [5])

$$\int_0^{\mu[x(T)]} Z^2(t) e^{-\frac{2}{x(t)}t} dt = \int_0^T Z^2(t) dt,$$

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where each admissible function $\mu(y)$ generates a solution

$$y = \varphi_\mu(T) = \varphi(T); \quad \mu(y) \geq 7y \ln y.$$

Remark 1. The main reason to introduce Jacob's ladders in [2] lies in the following: the Hardy-Littlewood integral (1918)

$$\int_0^T \left| \zeta \left(\frac{1}{2} + it \right) \right|^2 dt$$

has – in addition to the Hardy-Littlewood (and other similar) expression possessing unbounded errors at $T \rightarrow \infty$ – the following infinite set of almost exact expressions

$$(1.4) \quad \int_0^T \left| \zeta \left(\frac{1}{2} + it \right) \right|^2 dt = \varphi_1(T) \ln \varphi_1(T) + (c - \ln 2\pi) \varphi_1(T) + c_0 + \mathcal{O} \left(\frac{\ln T}{T} \right), \quad T \rightarrow \infty,$$

where c is the Euler's constant, and c_0 is the constant from the Titchmarsh-Kober-Atkinson formula (see [7], p. 141).

Remark 2. Simultaneously with (1.4) we have proved that the following transcendental equation

$$\int_0^T \left| \zeta \left(\frac{1}{2} + it \right) \right|^2 dt = V(T) \ln V(T) + (c - \ln 2\pi) V(T) + c_0$$

has an infinite set of asymptotic solutions

$$V(T) = \varphi_1(T), \quad T \rightarrow \infty.$$

Remark 3. The Jacob's ladder $\varphi_1(T)$ can be interpreted by our formula (see [2], (6.2))

$$(1.5) \quad T - \varphi_1(T) \sim (1 - c)\pi(T); \quad \pi(T) \sim \frac{T}{\ln T}, \quad T \rightarrow \infty,$$

where $\pi(T)$ is the prime-counting function, as an asymptotic complement function to the function

$$(1 - c)\pi(T)$$

in the sense

$$\varphi_1(T) + (1 - c)\pi(T) \sim T, \quad T \rightarrow \infty.$$

1.2. In the paper [3] we have proved that the following continuum set $S(T, 2l)$ of the systems

$$(1.6) \quad \left\{ |\tilde{Z}(t)|, |\tilde{Z}(t)| \cos \left[\frac{\pi}{l}(\varphi_1(t) - T) \right], |\tilde{Z}(t)| \sin \left[\frac{\pi}{l}(\varphi_1(t) - T) \right], \dots \right. \\ \left. |\tilde{Z}(t)| \cos \left[\frac{\pi}{l}n(\varphi_1(t) - T) \right], |\tilde{Z}(t)| \sin \left[\frac{\pi}{l}n(\varphi_1(t) - T) \right], \dots \right\}, \\ t \in \left[T, \overbrace{T+2l}^1 \right]; \\ \varphi_1 \left\{ \left[T, \overbrace{T+2l}^1 \right] \right\} = [T, T+2l]$$

is the set of orthogonal system on the segment

$$\left[T, \overbrace{T+2l}^1 \right]$$

for all

$$T \geq T_0[\varphi_1], \quad 2l \in \left(0, \frac{T}{\ln T}\right].$$

Next, in the paper [4] we have constructed corresponding continuum set of orthogonal systems generated by Jacobi's polynomials.

In this paper we give essential generalization of above mentioned. Namely, to every fixed L_2 -orthogonal system

$$\{f_n(t)\}_{n=1}^\infty, \quad t \in [0, 2l]$$

we assign continuum set of L_2 -orthogonal systems

$$\begin{aligned} & \{F_n(t; T, k, l)\}_{n=1}^\infty, \quad t \in [T, \widehat{T+2l}^k], \quad T \rightarrow \infty, \quad k = 1, \dots, k_0, \\ & l = o\left(\frac{T}{\ln T}\right); \\ & \varphi_1 \left\{ \widehat{[T, T+2l]}^k \right\} = \left[\widehat{T}^{k-1}, \widehat{T+2l}^{k-1} \right], \end{aligned}$$

where $k_0 \in \mathbb{N}$ is an arbitrary fixed number.

2. RESULT

2.1. Let us remind that (see [6])

$$\varphi_1^r(t) : \varphi_1^0(t) = t, \quad \varphi_1^1(t) = \varphi_1(t), \quad \varphi_1^2(t) = \varphi_1(\varphi_1(t)), \dots$$

The following Theorem holds true.

Theorem. For every fixed L_2 -orthogonal system

$$(2.1) \quad \{f_n(t)\}_{n=1}^\infty, \quad t \in [0, 2l], \quad l = o\left(\frac{T}{\ln T}\right), \quad T \rightarrow \infty$$

there is continuum set of L_2 -orthogonal systems

$$(2.2) \quad \begin{aligned} & \{F_n(t; T, k, l)\}_{n=1}^\infty = \\ & = \left\{ f_n(\varphi_1^k(t) - T) \prod_{r=0}^{k-1} \left| \tilde{Z}[\varphi_1^r(t)] \right| \right\}_{n=1}^\infty, \quad t \in [\widehat{T}^k, \widehat{T+2l}^k], \end{aligned}$$

where

$$(2.3) \quad \begin{aligned} & \varphi_1 \left\{ \widehat{[T, T+2l]}^k \right\} = \left[\widehat{T}^{k-1}, \widehat{T+2l}^{k-1} \right], \quad k = 1, \dots, k_0, \\ & \left[\widehat{\overset{\circ}{T}}, \widehat{\overset{\circ}{T+2l}} \right] = [T, T+2l], \quad T \rightarrow \infty \end{aligned}$$

and $k_0 \in \mathbb{N}$ is arbitrary number, i. e. the following formula is valid

$$(2.4) \quad \begin{aligned} & \int_{\widehat{T}^k}^{\widehat{T+2l}^k} f_m(\varphi_1^k(t) - T) f_n(\varphi_1^k(t) - T) \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_1^r(t)] dt = \\ & = \begin{cases} 0 & , \quad m \neq n, \\ A_n & , \quad m = n, \end{cases} \quad A_n = \int_0^{2l} f_n^2(t) dt. \end{aligned}$$

Next, we have the following properties

$$l = o\left(\frac{T}{\ln T}\right) \Rightarrow$$

$$(2.5) \quad |\widehat{[T, T+2l]}^k| = \widehat{T+2l}^k - \widehat{T}^k = o\left(\frac{T}{\ln T}\right),$$

$$(2.6) \quad |\widehat{[T+2l, T]}^{k-1}| = \widehat{T}^k - \widehat{T+2l}^{k-1} \sim (1-c)\pi(T); \quad \pi(T) \sim \frac{T}{\ln T},$$

$$(2.7) \quad [T, T+2l] \prec \widehat{[T, T+2l]}^1 \prec \dots \prec \widehat{[T, T+2l]}^k \prec \dots,$$

where $\pi(T)$ stands for the prime-counting function.

2.2.

Remark 4. We obtain from (2.2) by (1.3) that

$$(2.8) \quad F_n(t; T, k, l) = f_n(\varphi_1^k(t) - T) \prod_{r=0}^{k-1} \frac{|\zeta\left(\frac{1}{2} + i\varphi_1^r(t)\right)|}{\sqrt{\omega[\varphi_1^r(t)]}},$$

i. e. our formula (2.8) shows direct connection between the Riemann function

$$\zeta\left(\frac{1}{2} + it\right)$$

and an arbitrary L_2 -orthogonal system

$$\{f_n(t)\}_{n=1}^\infty, \quad t \in [0, 2l].$$

Remark 5. Asymptotic behavior of the disconnected set (see (2.6), (2.7))

$$(2.9) \quad \Delta(T, k, l) = \bigcup_{r=0}^k \widehat{[T, T+2l]}^r$$

is as follows: if $T \rightarrow \infty$, then the components of the set (2.9) recedes unboundedly each from other and all together are receding to infinity. Hence, if $T \rightarrow \infty$ the set (2.9) behaves as one dimensional Friedmann-Hubble expanding universe.

2.3. Since (see (2.3))

$$\begin{aligned} t \in \widehat{[T, T+2l]}^k &\Rightarrow \\ \varphi_1(t) \in [\varphi_1(T), \varphi_1(T+2l)] &= [\widehat{T}^{k-1}, \widehat{T+2l}^{k-1}] \Rightarrow \\ \varphi_1^2(t) \in [\varphi_1(\widehat{T}^{k-1}), \varphi_1(\widehat{T+2l}^{k-1})] &= [\widehat{T}^{k-2}, \widehat{T+2l}^{k-2}] \Rightarrow \\ &\vdots \end{aligned}$$

we point-out the following

Property 1. If

$$t \in [T, \widehat{T+2l}^k], \quad k = 1, \dots, k_0$$

then

$$(2.10) \quad \varphi_1^r(t) \in [T, \widehat{T+2l}^{k-r}], \quad r = 0, 1, \dots, k$$

holds true for the arguments of the functions (see (2.2), (2.8))

$$f_n(\varphi_1^k(t) - T), |\tilde{Z}[\varphi_1^r(t)]|, \omega[\varphi_1^r(t)], \left| \zeta \left(\frac{1}{2} + i\varphi_1^r(t) \right) \right|.$$

3. EXAMPLES

3.1. For the classical Fourier orthogonal system

$$(3.1) \quad \left\{ 1, \cos \frac{\pi t}{l}, \sin \frac{\pi t}{l}, \dots, \cos \frac{\pi n t}{l}, \sin \frac{\pi n t}{l}, \dots \right\}$$

$$t \in [0, 2l], \quad l = o\left(\frac{T}{\ln T}\right)$$

we have as corresponding (see (2.2), (2.8)) continuous set of orthogonal systems the following

$$(3.2) \quad \left\{ \prod_{r=0}^{k-1} \frac{|\zeta(\frac{1}{2} + i\varphi_1^r(t))|}{\sqrt{\omega[\varphi_1^r(t)]}}, \dots, \right.$$

$$\left(\prod_{r=0}^{k-1} \frac{|\zeta(\frac{1}{2} + i\varphi_1^r(t))|}{\sqrt{\omega[\varphi_1^r(t)]}} \right) \cos \left(\frac{\pi}{l} n(\varphi_1^k(t) - T) \right),$$

$$\left(\prod_{r=0}^{k-1} \frac{|\zeta(\frac{1}{2} + i\varphi_1^r(t))|}{\sqrt{\omega[\varphi_1^r(t)]}} \right) \sin \left(\frac{\pi}{l} n(\varphi_1^k(t) - T) \right), \dots \left. \right\},$$

$$t \in [T, \widehat{T+2l}^k], \quad k = 1, \dots, k_0, \quad T \rightarrow \infty,$$

and, for example,

$$k_0 = S = 10^{10^{34}}, \quad S^S, \dots$$

where S is the Skeewes constant.

3.2. For the system of Jacobi's functions

$$(3.3) \quad \sqrt{(1-t)^\alpha(1+t)^\beta} P_n^{(\alpha, \beta)}(t), \quad t \in [-1, 1], \quad n = 0, 1, 2, \dots; \quad \alpha, \beta > -1$$

generated by the Jacobi's polynomials $P_n^{(\alpha, \beta)}$ we have that

$$(3.4) \quad \int_{-1}^1 (1-t)^\alpha (1+t)^\beta P_n^{(\alpha, \beta)}(t) P_m^{(\alpha, \beta)}(t) dt = 0, \quad m \neq n,$$

$$\int_{-1}^1 (1-t)^\alpha (1+t)^\beta [P_n^{(\alpha, \beta)}(t)]^2 dt =$$

$$= \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)} = A_n(\alpha, \beta).$$

Next, the substitution

$$x = t - T - 1$$

in (3.3) yields (see (3.4)) the formulae

$$\int_T^{T+2} (2+T-t)^\alpha (t-T)^\beta P_m^{(\alpha,\beta)}(t-T-1) P_n^{(\alpha,\beta)}(t-T-1) dt = 0, \quad m \neq n, \dots$$

Consequently, the following continuum set (for each fixed pair $\alpha, \beta > -1$) of orthogonal systems

$$\left\{ P_n^{(\alpha,\beta)}(t-T-1) \sqrt{(T+2-\varphi_1^k(t))^\alpha (\varphi_1^k(t)-T)^\beta} \times \prod_{r=0}^{k-1} \frac{|\zeta(\frac{1}{2} + i\varphi_1^r(t))|}{\sqrt{\omega[\varphi_1^r(t)]}} \right\}_{n=0}^\infty, \\ t \in [T, \widehat{T+2}^k], \quad T \rightarrow \infty, \quad k = 1, \dots, k_0.$$

corresponds to the Jacobi's orthogonal system (3.3) (see (2.2), (2.8)).

3.3. For the system of Bessel's functions

$$(3.5) \quad \left\{ \sqrt{t} J_n \left(\frac{\mu_m^{(n)}}{2l} t \right) \right\}_{m=1}^\infty, \quad t \in [0, 2l]$$

generated by Bessel's function $J_n(t)$ we have that

$$\int_0^{2l} t J_n \left(\frac{\mu_{m_1}^{(n)}}{2l} t \right) J_n \left(\frac{\mu_{m_2}^{(n)}}{2l} t \right) dt = 0, \quad m_1 \neq m_2, \\ \int_0^{2l} t \left[J_n \left(\frac{\mu_m^{(n)}}{2l} t \right) \right]^2 dt = 2l^2 [J'_n(\mu_m^{(n)})]^2,$$

where

$$\{\mu_m^{(n)}\}_{m=1}^\infty$$

is the sequence of the roots of equation

$$J_n(\mu) = 0.$$

Consequently, the following continuum set (for each fixed n) of orthogonal systems

$$\left\{ J_n \left(\frac{\mu_m^{(n)}}{2l} (\varphi_1^k(t) - T) \right) \sqrt{\varphi_1^k(t) - T} \prod_{r=0}^{k-1} \frac{|\zeta(\frac{1}{2} + i\varphi_1^r(t))|}{\sqrt{\omega[\varphi_1^r(t)]}} \right\}_{m=1}^\infty, \\ t \in [T, \widehat{T+2l}^k], \quad T \rightarrow \infty, \quad k = 1, \dots, k_0.$$

corresponds to the Bessel orthogonal system (3.5) (see (2.2), (2.8)).

4. FORMULA (2.4) AS A RESOURCE OF NEW INTEGRALS CONTAINING MULTIPLES OF $|\zeta|^2$

We consider the formula (see (2.4), (2.8))

$$(4.1) \quad \int_T^{\widehat{T+2l}^k} f_n^2(\varphi_1^k(t) - T) \prod_{r=0}^{k-1} \frac{|\zeta(\frac{1}{2} + i\varphi_1^r(t))|^2}{\omega[\varphi_1^r(t)]} dt = A_n, \\ A_n = \int_0^{2l} f_n^2(t) dt, \quad n = 1, 2, \dots$$

4.1. Let (see (2.9))

$$t \in \Delta^0(T, k, l) = \bigcup_{r=0}^k (\widehat{T, T+2l}^r), \quad k = 1, \dots, k_0.$$

Of course

$$\Delta^0(T, k, l) \subset [\widehat{T, T+2l}^k],$$

and (see (2.5) – (2.7))

$$\begin{aligned} (4.2) \quad |\widehat{T, T+2l}^k| &= \sum_{r=0}^k |\widehat{T, T+2l}^r| + \sum_{r=1}^k |\widehat{T+2l, T}^r| = \\ &= (k+1)O\left(\frac{T}{\ln T}\right) + kO\left(\frac{T}{\ln T}\right) = \\ &= O\left(\frac{T}{\ln T}\right), \quad k = 1, \dots, k_0. \end{aligned}$$

Thus, we have the following: if

$$t \in [\widehat{T, T+2l}^k],$$

then (see (4.2))

$$\ln t = \ln(t - T + T) = \ln T + \ln\left(1 + \frac{t - T}{T}\right) = \ln T + O\left(\frac{1}{\ln T}\right),$$

i. e.

$$(4.3) \quad \ln t \sim \ln T, \quad \forall t \in (\widehat{T, T+2l}^k), \quad k = 1, \dots, k_0.$$

4.2. It is sufficient to use, for example, the formula (4.1) in the case (see (3.1))

$$f(t) = 1, \Rightarrow A_1 = 2l,$$

i. e.

$$(4.4) \quad \int_{\widehat{T, T+2l}^k} \prod_{r=0}^{k-1} \frac{|\zeta(\frac{1}{2} + i\varphi_1^r(t))|^2}{\omega[\varphi_1^r(t)]} dt = 2l.$$

Next, we obtain from (4.4) (see (1.3), (4.3)) by the mean-value theorem that

$$(4.5) \quad \int_{\widehat{T, T+2l}^k} \prod_{r=0}^{k-1} \left| \zeta\left(\frac{1}{2} + i\varphi_1^r(t)\right) \right|^2 dt \sim 2l \ln^k T, \quad T \rightarrow \infty.$$

Consequently, we obtain from (4.5) in the case

$$2l = \frac{\Omega}{\ln^k T} = o\left(\frac{T}{\ln T}\right), \quad \Omega > 0$$

the following

Corollary.

$$(4.6) \quad \int_T^{\overbrace{T+\Omega \ln^{-k} T}^k} \prod_{r=0}^{k-1} \left| \zeta \left(\frac{1}{2} + i\varphi_1^r(t) \right) \right|^2 dt \sim \Omega, \quad T \rightarrow \infty,$$

where

$$0 < \Omega = o(T \ln^{k-1} T), \quad k = 1, \dots, k_0.$$

Remark 6. Let us notice explicitly that nor the first two formulae (see (4.6), $k = 1, 2$; $\Omega = 1$)

$$(4.7) \quad \int_{\frac{1}{T}}^{\overbrace{T+\ln^{-1} T}^1} \left| \zeta \left(\frac{1}{2} + it \right) \right|^2 dt \sim 1,$$

$$\int_{\frac{1}{T}}^{\overbrace{T+\ln^{-1} T}^1} \left| \zeta \left(\frac{1}{2} + it \right) \right|^2 \left| \zeta \left(\frac{1}{2} + i\varphi_1(t) \right) \right|^2 dt \sim 1, \quad T \rightarrow \infty$$

are not accessible by the current methods in the theory of the Riemann zeta-function.

Remark 7. The first formula in (4.7) gives us the answer to the question about a form of segments for which the following

$$[a(T), b(T)] \rightarrow \int_{a(T)}^{b(T)} \left| \zeta \left(\frac{1}{2} + it \right) \right|^2 dt \sim 1, \quad T \rightarrow \infty$$

holds true. Namely, corresponding segments are as follows

$$[a(T), b(T)] = \left[\frac{1}{T}, \overbrace{T + \frac{1}{\ln T}}^1 \right] \leftarrow \left[T, T + \frac{1}{\ln T} \right].$$

5. FIRST LEMMAS

5.1. The sequence

$$(5.1) \quad \{T\}_{k=1}^{\infty}$$

is defined by the formula (comp. (2.3))

$$(5.2) \quad \varphi_1^k(T) = \frac{k-1}{T}, \quad k = 1, \dots, k_0, \quad \frac{0}{T} = T$$

for every $T \geq T_0[\varphi_1]$, where $k_0 \in \mathbb{N}$ is an arbitrary fixed number. Since the function

$$\varphi_1(t), \quad t \rightarrow \infty$$

increases to ∞ , then we have from (5.2) that

$$\left\{ \frac{k-1}{T} \rightarrow \infty \right\} \Leftrightarrow \left\{ \frac{k}{T} \rightarrow \infty \right\},$$

i. e.

$$(5.3) \quad \{T \rightarrow \infty\} \Leftrightarrow \left\{ \frac{k}{T} \rightarrow \infty \right\}.$$

Next, we have (see (1.5), (5.2), (5.3)) that

$$(5.4) \quad \frac{k}{T} - \frac{k-1}{T} \sim (1-c) \frac{\frac{k}{T}}{\ln T} \Rightarrow 1 - \frac{k-1}{\frac{k}{T}} \sim \frac{1-c}{\ln T},$$

i. e.

$$\frac{k-1}{T} \sim \frac{k}{T},$$

and, consequently,

$$(5.5) \quad \frac{k}{T} = \{1 + o(1)\}T, \quad T \rightarrow \infty, \quad k = 1, \dots, k_0.$$

Since

$$\frac{\frac{k}{T}}{\ln T} \sim \frac{T}{\ln T},$$

(see (5.5)) then we have (see 5.4)) for the sequence (5.1) that

$$(5.6) \quad \frac{k}{T} - \frac{k-1}{T} \sim (1-c) \frac{T}{\ln T}, \quad T \rightarrow \infty, \quad k = 1, \dots, k_0.$$

Consequently, we have

$$(5.7) \quad T < \frac{1}{T} < \dots < \frac{k_0}{T}$$

and, of course, (see (2.1))

$$(5.8) \quad T + H < \widehat{T + H}^1 < \dots < \widehat{T + H}^{k_0}, \quad 0 < H = o\left(\frac{T}{\ln T}\right).$$

5.2. The following lemma holds true.

Lemma 1.

$$(5.9) \quad H = o\left(\frac{T}{\ln T}\right) \Rightarrow$$

$$|[\widehat{T, \widehat{T + H}^k}]| = \widehat{T + H}^k - T = o\left(\frac{T}{\ln T}\right), \quad T \rightarrow \infty, \quad k = 1, \dots, k_0,$$

i. e. (2.5) holds true.

Proof. First of all, it follows from (5.6) that

$$\frac{k}{T} - T \sim (1-c)k \frac{T}{\ln T},$$

i. e.

$$(5.10) \quad \frac{k}{T} - T = \{1 + o_1(1)\}(1-c)k \frac{T}{\ln T}$$

and, simultaneously (see (5.8))

$$(5.11) \quad \widehat{T + H}^k - (T + H) = \{1 + o_2(1)\}(1-c)k \frac{T}{\ln T}.$$

Then we have (see (5.9) – (5.11)) that

$$\begin{aligned}
 0 < \overbrace{T+H}^k - \overset{k}{T} &= H + [o_2(1) - o_1(1)](1-c)k \frac{T}{\ln T} = \\
 &= H + [o_4(1) - o_3(1)] \frac{T}{\ln T} = \\
 &= o\left(\frac{T}{\ln T}\right) + o(1) \frac{T}{\ln T} = \\
 &= o\left(\frac{T}{\ln T}\right), \quad T \rightarrow \infty.
 \end{aligned}$$

□

5.3. Next, the following lemma holds true

Lemma 2.

$$\begin{aligned}
 (5.12) \quad H &= o\left(\frac{T}{\ln T}\right) \Rightarrow \\
 \overset{k}{T} - \overbrace{T+H}^{k-1} &\sim (1-c) \frac{T}{\ln T}, \quad T \rightarrow \infty, \quad k = 1, \dots, k_0,
 \end{aligned}$$

i. e. (2.6) holds true.

Proof. We have from (5.6) by (5.8), (5.9) that

$$\begin{aligned}
 \overset{k}{T} - \overbrace{T+H}^{k-1} + \overbrace{T+H}^{k-1} - \overset{k-1}{\widehat{T}} &\sim (1-c) \frac{T}{\ln T}, \\
 \overset{k}{T} - \overbrace{T+H}^{k-1} &\sim (1-c) \frac{T}{\ln T} - (\overbrace{T+H}^{k-1} - \overset{k-1}{\widehat{T}}) \sim \\
 &\sim (1-c) \frac{T}{\ln T} + o\left(\frac{T}{\ln T}\right) \sim \\
 &\sim (1-c) \frac{T}{\ln T}, \quad T \rightarrow \infty, \quad k = 1, \dots, k_0.
 \end{aligned}$$

□

Remark 8. We have (see (5.12)) that

$$(5.13) \quad [T, T+H] \prec [\overset{1}{T}, \overbrace{T+H}^1] \prec \dots \prec [\overset{k_0}{T}, \overbrace{T+H}^{k_0}],$$

i. e. (2.7) holds true.

6. REVERSE ITERATIONS

6.1. First of all, we have (see (2.3), (5.2)) that

$$(6.1) \quad \varphi_1(\overset{k}{T}) = \overset{k-1}{T} \Rightarrow \dots \Rightarrow \varphi_1^k(\overset{k}{T}) = T, \quad k = 1, \dots, k_0.$$

Since

$$(6.2) \quad \varphi_1(\overset{1}{T}) = T \Rightarrow \overset{1}{T} = \varphi_1^{-1}(T)$$

then we may use the inverse function

$$\varphi_1^{-1}(T)$$

to generate reverse iterations. We have (see (6.2)) that

$$(6.3) \quad \begin{aligned} \varphi_1(\overset{2}{T}) = \overset{1}{T} &\Rightarrow \overset{2}{T} = \varphi_1^{-1}(\overset{1}{T}) = \varphi_1^{-1}(\varphi_1^{-1}(T)) = \varphi_1^{-2}(T), \\ &\vdots \\ \overset{k}{T} &= \varphi_1^{-k}(T), \quad k = 1, \dots, k_0. \end{aligned}$$

Of course, we have (see (6.1), (6.3)) that

$$\varphi_1^k(\overset{k}{T}) = \varphi_1^k(\varphi_1^{-k}(T)) = \varphi_1^0(T) = T.$$

6.2. Next, the following holds true.

Property 2. If

$$t \in [\varphi_1^{-k}(T), \varphi_1^{-k}(T+H)]$$

then (see (6.3))

$$\begin{aligned} \varphi_1^r(t) &\in [\varphi_1^r(\varphi_1^{-k}(T)), \varphi_1^r(\varphi_1^{-k}(T+H))] = \\ &= [\varphi_1^{r-k}(T), \varphi_1^{r-k}(T+H)] \end{aligned}$$

(comp. (2.10)), i. e.

$$(6.4) \quad \begin{aligned} \varphi_1^0(t) &= t \in [\varphi_1^{-k}(T), \varphi_1^{-k}(T+H)] = [\overset{k}{T}, \overset{k}{T+H}], \\ \varphi_1^1(t) &\in [\varphi_1^{-k+1}(T), \varphi_1^{-k+1}(T+H)] = [\overset{k-1}{T}, \overset{k-1}{T+H}], \\ &\vdots \\ \varphi_1^{k-1}(t) &\in [\varphi_1^{-1}(T), \varphi_1^{-1}(T+H)] = [\overset{1}{T}, \overset{1}{T+H}], \\ \varphi_1^k(t) &\in [\varphi_1^0(T), \varphi_1^0(T+H)] = [T, T+H]. \end{aligned}$$

Remark 9. Of course, the following holds true (see (5.13), (6.4))

$$(6.5) \quad \begin{aligned} [T, T+H] &\prec [\varphi_1^{-1}(T), \varphi_1^{-1}(T+H)] \prec \dots \\ &\prec [\varphi_1^{-k}(T), \varphi_1^{-k}(T+H)], \quad k = 1, \dots, k_0. \end{aligned}$$

Let us remind for comparison that (see [6], (2.5), Remark 7)

$$\begin{aligned} [T, T+H] &\succ [\varphi_1(T), \varphi_1(T+H)] \succ \dots \succ \\ &\succ [\varphi_1^k(T), \varphi_1^k(T+H)], \end{aligned}$$

where the direct iteration $\varphi_1^k(T)$ is generated by the function $\varphi_1(T)$.

Remark 10. The first two reverse iterations of the segments

$$[T, T + \frac{1}{\ln T}], \quad [T, T + \frac{1}{\ln^2 T}]$$

are included in integral formulae (4.7).

7. MAIN LEMMA

7.1. The following Lemma holds true.

Lemma 3. *If*

$$(7.1) \quad H = o\left(\frac{T}{\ln T}\right), \quad T \rightarrow \infty$$

then for every Lebesgue-integrable function

$$f(t), \quad t \in [T, T + H]$$

we have that

$$(7.2) \quad \int_T^{T+H} f(t) dt = \int_T^{\widehat{T+H}^k} f[\varphi_1^k(t)] \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_1^r(t)] dt,$$

$$T \rightarrow \infty, \quad k = 1, \dots, k_0; \quad \varphi_1^0(t) = t,$$

where $k_0 \in \mathbb{N}$ is arbitrary fixed number.

Proof. In our paper [5], (9.2), (9.5) we have proved the following lemma: if (comp. (2.3))

$$\varphi_1\{\widehat{[T, T+H]}^1\} = [T, T+H]$$

then for every Lebesgue-integrable function

$$f(t), \quad t \in [T, T + H]$$

we have (comp. (1.2), (1.3))

$$(7.3) \quad \int_T^{T+H} f(t) dt = \int_T^{\widehat{T+H}^1} f[\varphi_1(t)] \tilde{Z}^2(t) dt,$$

$$T \geq T_0[\varphi_1], \quad H \in \left(0, \frac{T}{\ln T}\right].$$

Another form of (7.3) is expressed by the formula (see (1.3))

$$(7.4) \quad \int_T^{T+H} f(t) dt = \int_T^{\widehat{T+H}^1} f[\varphi_1(t)] \frac{Z^2(t)}{\omega(t)} dt.$$

Now, the repeated application of the formula (7.3) (see (5.9)) gives the following: if

$$H = o\left(\frac{T}{\ln T}\right)$$

then

$$\begin{aligned}
\int_T^{T+H} f(t)dt &= \int_T^{\widehat{T+H}^1} f[\varphi_1(t)]\tilde{Z}^2(t)dt = \\
&= \int_T^{\widehat{T+H}^2} f[\varphi_1^2(t)]\tilde{Z}^2[\varphi_1(t)]\tilde{Z}^2(t)dt = \dots = \\
&= \int_T^{\widehat{T+H}^k} f[\varphi_1^k(t)] \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_1^r(t)]dt,
\end{aligned}$$

that is exactly (7.2). \square

7.2.

Remark 11. The formula (7.2) can be expressed as follows (see (6.3))

$$\int_T^{T+H} f(t)dt = \int_{\varphi_1^{-k}(T)}^{\varphi_1^{-k}(T+H)} f[\varphi_1^k(t)] \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_1^r(t)]dt, \quad T \rightarrow \infty.$$

8. PROOF OF THEOREM

Let

$$\{f_n(t)\}_{n=1}^\infty, \quad t \in [0, 2l], \quad l = o\left(\frac{T}{\ln T}\right)$$

be arbitrary fixed L_2 -orthogonal system, i. e.

$$(8.1) \quad \int_0^{2l} f_m(t)f_n(t)dt = \begin{cases} 0 & , \quad m \neq n, \\ A_n & , \quad m = n, \end{cases} \quad A_n = \int_0^{2l} f_n^2(t)dt.$$

Then we have for corresponding system (2.2) by (7.2), (8.1) that

$$\begin{aligned}
f(t) &\longrightarrow f_m(\varphi_1^k(t) - T)f_n(\varphi_1^k(t) - T), \\
\int_T^{\widehat{T+2l}^k} f_m(\varphi_1^k(t) - T)f_n(\varphi_1^k(t) - T) \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_1^r(t)]dt &= \\
&= \int_T^{T+2l} f_m(t - T)f_n(t - T)dt = \\
&= \int_0^{2l} f_m(t)f_n(t)dt = \begin{cases} 0 & , \quad m \neq n, \\ A_n & , \quad m = n, \end{cases}
\end{aligned}$$

i. e. (2.3) holds true. Finally, the properties (2.5) – (2.7) follows from (5.9), (5.12), (5.13).

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